ON TRANSFER IN BOUNDED COHOMOLOGY

INDIRA CHATTERJI AND AND GUIDO MISLIN

Abstract. We define a transfer map in the setting of bounded cohomology with certain metric G-module coefficients. As an application, we extend a theorem on the comparison map from Borel-bounded to Borel cohomology (cf. [1]), to cover the case of Lie groups with finitely many connected components.

1. Introduction

The groups G we are working with are always assumed to be separable, locally compact, with a topology given by a complete metric. Borel cohomology groups $H^*_B(G, A)$ for G with coefficients in a polish G-module A where introduced by Moore [3]. In case A is equipped with a metric, one can define the Borel-bounded cohomology groups $H^*_{Bb}(G, A)$ and study the comparison map from Borel-bounded to Borel cohomology. The case of a connected Lie group G and $A = \mathbb{Z}$ with trivial G-action was dealt with in [1]. The purpose of this note is to define a transfer map in the H^*_B and H^*_{Bb} setting, with respect to a closed subgroup $H < G$ of finite index. We will make use of our transfer map to extend some results proved in [1] for connected Lie groups, to cover the case of groups with finitely many connected components (*virtually connected* groups).

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2. The Borel (bounded) setup

Through this note, G will denote a separable, locally compact topological group, with a topology given by a complete metric.

Definition 2.1. We denote by $P(G)$ the category of G-modules which are separable, completely metrizable with an isometric G-action (*Polish G-modules*); morphisms are continuous G-maps.

When dealing with bounded cohomology, we need to fix a metric on our coefficient modules. Let $M \in P(G)$. Let m_1 and m_2 be two

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complete metrics on M (compatible with the given topology) with respect to which the G-action is isometric. We say that m_1 and m_2 are *bounded equivalent* (b-equivalent for short), if they define the same bounded subsets in M.

Definition 2.2. We denote by $mP(G)$ the category of G-modules M in $P(G)$, equipped with a preferred b-equivalence class $[m]$ of metrics. Morphisms in $mP(G)$ are continuous G-maps which map bounded subsets to bounded subsets.

If $H < G$ is a closed normal subgroup of finite index and M is a finitely generated $\mathbb{Z}G$-module with trivial H-action, then the word metric m_S on the underlying abelian group of M, with respect to a finite G-stable generating set $S < M$ defines a b-equivalence class $[m_S]$ of complete metrics on M, independent of the particular S chosen. In this way we obtain a fully faithful functor

$$\mathrm{fgMod}(G/H) \to mP(G)\,,$$

with $\mathrm{fgMod}(G/H)$ denoting the category of finitely generated G/H-modules.

There is an obvious forgetful functor $mP(G) \to P(G)$. In this way, we may view an $M \in \mathrm{fgMod}(G/H)$ as an object in $mP(G)$, respectively $P(G)$.

For $M \in P(G)$, a map $f : G^{n+1} \to M$ is called *Borel*, if it is Borel measurable with respect to the σ-algebras associated to the topological spaces G^{n+1} and M respectively; in case M is in $mP(G)$, f is called *bounded*, if its range is a bounded subset of M; f is called *G-equivariant*, if $f(xg_0, \cdots, xg_n) = xf(g_0, \cdots, g_n)$. We write

$$C^n_B(G, M) = \mathrm{map}^G_B(G^{n+1}, M) \text{ and } C^n_{Bb}(G, M) = \mathrm{map}^G_{Bb}(G^{n+1}, M)$$

for the abelian group of G-equivariant Borel (respectively Borel bounded) maps $G^{n+1} \to M$. With the usual differentials, this defines cochain complexes $C^*_B(G, M)$ and $C^*_{Bb}(G, M)$ whose respective cohomologies,

$$H^*_B(G, M) \text{ and } H^*_{Bb}(G, M)$$

is the *Borel* (respectively *Borel bounded*) cohomology of G with coefficients in M (for Borel cohomology with coefficients in $P(G)$, see Moore [3]; Borel bounded cohomology with Banach G-module coefficients has been dealt with in [2]).

Remark 2.3. Let G denote a virtually connected Lie group with connected component G_0. For $M \in P(G)$ with underlying topology of M discrete, the G-action on M factors through G/G_0 so that M is a G/G_0-module. Note also that, because G is virtually connected

$\pi_1(BG) = \pi_0(G) = G/G_0$ is a finite group. According to Wigner [5, Theorem 4], see also Moore [3, Section 7, (1)], one has for M discrete and countable a natural isomorphism

$$H^*_B(G, M) \cong H^*(BG, M)$$

where $H^*(BG, M)$ denotes the singular cohomology of BG with local coefficients in the $\pi_1(BG)$-module M. Similarly, for V a continuous finite-dimensional $\mathbb{R}$-representation of G, one has a natural isomorphism

$$H^*_B(G, V) \cong H^*_c(G, V)\,,$$

with $H^*_c(G, V)$ denoting the continuous cohomology of G [3, Section 7, (2)].

Let $H < G$ be a closed subgroup of finite index and let $M \in P(H)$. We write $\pi : G \to G/H$ for the projection and we fix a section $\sigma : G/H \to G$ satisfying $\sigma(1 \cdot H) = 1 \in G$. There is a continuous retraction $\rho : G \to H$ given by $\rho(g) = g \cdot \sigma(\pi(g^{-1}))$. We define the coinduced module $\mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}} M \in P(G)$ as follows: the underlying abelian group consist of all continuous functions $f : G \to M$ such that $f(gh) = h^{-1}f(g)$ for all $(g, h) \in G \times H$. This defines a coinduction functor

$$\mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}} : P(H) \to P(G)\,.$$

If $M \in mP(G)$, that is, M is equipped with d_M a compatible preferred metric, we observe that the range of $d_M(f(?), 0) : G \to \mathbb{R}$ is compact, because $d_M(f(gh), 0) = d_M(f(g), 0)$ for $h \in H$. We define a metric on $\mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}} M$, using the sup-metric $||f|| = \sup_G(d_M(f(g), 0))$. The isometric G-module structure on $\mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}} M$ is given by $(x \cdot f)(g) = f(x^{-1}g)$. Hence the functor $\mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}}$ defines a coinduction functor

$$\mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}} : mP(H) \to mP(G)\,.$$

It also follows that one has a natural isomorphism

$$(\mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}} M)^G \xrightarrow{\cong} M^H\,, f \mapsto f(1)\,.$$

The following partial *Eckmann-Shapiro* Lemma holds in this situation.

Lemma 2.4. *Let G be a separable, locally compact topological group, with a topology given by a complete metric and let $H < G$ a closed subgroup of finite index. Let $M \in mP(H)$. There is a natural diagram*

$$\begin{array}{ccc} H^*_{Bb}(G, \mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}} M) & \xrightarrow{\Theta^*_b} & H^*_{Bb}(H, M) \\ \downarrow & & \downarrow \\ H^*_B(G, \mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}} M) & \xrightarrow{\Theta^*} & H^*_B(H, M) \end{array}$$

and both maps Θ^* *and* Θ_b^* *are onto.*

Proof. The map of cochain complexes:

$$\Theta : C_B^*(G, \mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}} M) \to C_B^*(H, M)\,,$$

given on n-cochains by $\Theta(f)(h_0, \cdots, h_n) = f(h_0, \cdots, h_n)(1)$, restricts to a map Θ_b of bounded cochains. Moreover, the map

$$\Lambda : C_B^*(H, M) \to C_B^*(G, \mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}} M)$$

given by $\Lambda(f)(g_0, \cdots, g_n)(g) = f(\rho(g^{-1}g_0), \cdots, \rho(g^{-1}g_n))$ on n-cochains, satisfies

$$\begin{aligned}(\Theta \circ \Lambda)(f)(h_0, \cdots, h_n) &= \Lambda(f)(h_o, \cdots, h_n)(1) \\ &= f(\rho(h_0), \cdots, \rho(h_n)) = f(h_0, \cdots, h_n)\,,\end{aligned}$$

so that $\Theta \circ \Lambda = Id$ and therefore the induced map Θ^* and Θ_b^* are split surjections. □

Recall that the *radical* $\sqrt{G}$ of a (not necessarily connected) Lie group G is its maximal, connected, normal, solvable subgroup. A Lie groups is called *linear*, if it admits a faithful representation $G \to GL(n, \mathbb{R})$ for some n.

Corollary 2.5. *Let* G *be a virtually connected Lie group such that for every finitely generated* G/G_0*-module* M *the forgetful map* $H_{Bb}^*(G, M) \to H_B^*(G, M)$ *is surjective. Then the radical* $\sqrt{G}$ *of* G *is linear.*

Proof. Since $\mathrm{Coind}_{\mathrm{G_0}}^{\mathrm{G}} \mathbb{Z}$ is finitely generated as an abelian group, with the connected group G_0 acting trivially, it is a finitely generated G/G_0-module. We conclude from our assumption that

$$H_{Bb}^*(G, \mathrm{Coind}_{\mathrm{G_0}}^{\mathrm{G}} \mathbb{Z}) \to H_B^*(G, \mathrm{Coind}_{\mathrm{G_0}}^{\mathrm{G}} \mathbb{Z})$$

is surjective. From Lemma 2.4 with $H = G_0$ and $M = \mathbb{Z}$ we conclude that

$$H_{Bb}^*(G_0, \mathbb{Z}) \to H_B^*(G_0, \mathbb{Z})$$

is surjective too. Because G_0 is connected, we can apply Theorem 1.1 of [1] to conclude that $\sqrt{G_0}$ is linear, which completes the proof of the corollary, since $\sqrt{G_0} = \sqrt{G}$. □

3. Construction of the transfer map

Let G be a separable, locally compact topological group, with a topology given by a complete metric and let $H < G$ a closed subgroup of finite index. Let $M \in P(G)$. The map

$$\kappa : \mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}} M \to M, \quad f \mapsto \sum_{gH \in G/H} g \cdot f(g)$$

is well-defined, because for $h \in H$, $h^{-1}f(g) = f(gh)$ so that $gh \cdot f(gh) = g \cdot f(g)$. Moreover, κ is a G-equivariant map since for $x \in G$,

$$\begin{aligned} \kappa(x \cdot f) &= \sum_{G/H} g \cdot (xf)(g) = \sum_{G/H} g \cdot f(x^{-1}g) \\ &= x(\sum_{G/H} x^{-1}g \cdot f(x^{-1}g)) = x \cdot \kappa(f). \end{aligned}$$

We further note that, when working with a fixed compatible metric on M, then $||\kappa(f)|| \leq [G : H]||f||$. Hence κ maps bounded sets of $\mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}} M$ to bounded sets of M and induces

$$\kappa^* : H^*_{Bb}(G, \mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}} M) \to H^*_{Bb}(G, M)\,.$$

Definition 3.1. We define the *transfer map* in Borel cohomology for $M \in P(G)$ by

$$\mathrm{Tran} := \kappa^* \circ \Theta^* : H^*_B(H, M) \to H^*_B(G, \mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}} M) \to H^*_B(G, M)\,.$$

If $M \in mP(G)$, one similarly defines the *transfer map* in Borel bounded cohomology by

$$\mathrm{Tran} := \kappa^* \circ \Theta^*_b : H^*_{Bb}(H, M) \to H^*_{Bb}(G, \mathrm{Coind}_{\mathrm{H}}^{\mathrm{G}} M) \to H^*_{Bb}(G, M)\,.$$

Theorem 3.2. *Let G be a separable, locally compact topological group, with a topology given by a complete metric and let $H < G$ a closed subgroup of finite index. Let $M \in P(G)$ and denote by* $\mathrm{Res} : \mathrm{H}^*_{\mathrm{B}}(\mathrm{G}, \mathrm{M}) \to \mathrm{H}^*_{\mathrm{B}}(\mathrm{H}, \mathrm{M})$ *the restriction map. Then the composite map*

$$\mathrm{Tran} \circ \mathrm{Res} : \mathrm{H}^*_{\mathrm{B}}(\mathrm{G}, \mathrm{M}) \to \mathrm{H}^*_{\mathrm{B}}(\mathrm{G}, \mathrm{M})$$

is the multiplication by the index $[G : H]$. Moreover, if $M \in mP(G)$, there is a natural commutative diagram

$$\begin{array}{ccccc} H^*_{Bb}(G, M) & \xrightarrow{\mathrm{Res}} & H^*_{Bb}(G_0, M) & \xrightarrow{\mathrm{Tran}} & H^*_{Bb}(G, M) \\ \downarrow & & \downarrow & & \downarrow \\ H^*_B(G, M)) & \xrightarrow{\mathrm{Res}} & H^*_B(G_0, M) & \xrightarrow{\mathrm{Tran}} & H^*_B(G, M)\,. \end{array}$$

Proof. Consider $H^*_B(G, ?)$ as a cohomological functor on $P(G)$. In [3, Theorem 4] Moore proved that this functor is coeffaceable, that is, for $M \in P(G)$ there exist a closed embedding $M \to J(M)$ with $J(M) \in P(G)$ satisfying $H^*_B(G, J(M)) = 0$ for $* > 0$. It follows that any natural transformation $T^* : H^*_B(G, ?) \to H^*_B(G, ?)$ of functors on $P(G)$ is determined by $T^0 : H^0_B(G, ?) \to H^0_B(G, ?)$. Applying that remark to the natural transformation $T = \mathrm{Tran} \circ \mathrm{Res}$ one verifies for $M \in P(G)$ that in degree 0 the map $T^0 : H^0(G, M) \to H^0(G, M)$ corresponds to the map $M^G \to M^G$ given by $m \mapsto \sum_{gH \in G/H} gm = [G : H]m$. As a result, the composite map $\mathrm{Tran} \circ \mathrm{Res}$ is, as claimed, the multiplication by $[G : H]$, in all degrees. The commutativity of the diagram for the forgetful map is immediate from the definition of the transfer. □

4. Applications

If A is a finitely generated $\mathbb{Z}G$-module, with kernel of the G-action on A a normal subgroup of finite index in $H < G$, we can (as described earlier) consider A as an object in $mP(G)$, by choosing a word metric on A with respect to a G-stable finite generating set. Forming $A \otimes \mathbb{R}$, we can choose a G-invariant Euclidean metric on that finite dimensional $\mathbb{R}$-vector space, yielding an object $A \otimes \mathbb{R}$ in $mP(G)$ such that $A \to A \otimes \mathbb{R}$ is a morphism in $mP(G)$ (it maps bounded subsets to bounded subsets). We can also form $(A \otimes \mathbb{R}) / \operatorname{image}(A)$, which is topologically a finite dimensional torus, inheriting from $A \otimes \mathbb{R}$ a natural G-invariant metric, yielding an exact sequence

$$A \to A \otimes \mathbb{R} \to (A \otimes \mathbb{R}) / \operatorname{image}(A)$$

in $mP(G)$. We call this the *standard way* of viewing A, $A \otimes \mathbb{R}$ and $(A \otimes \mathbb{R}) / \operatorname{image}(A)$ as objects in $mP(G)$.

Lemma 4.1. *Let G be a separable, locally compact topological group, with a topology given by a complete metric and let $H < G$ a closed subgroup of finite index. Let A be a finitely generated $\mathbb{Z}G$-module with H acting trivially on A; view A and $A \otimes \mathbb{R}$ in the standard way as an object in $mP(G)$. If the natural map $H^*_B(G, A) \to H^*_B(G, A \otimes \mathbb{R})$ maps an element $x \in H^*_B(G, A)$ to an element $\overline{x} \in H^*_B(G, A \otimes \mathbb{R})$ which lies in the image of $H^*_{Bb}(G, A \otimes \mathbb{R}) \to H^*_B(G, A \otimes \mathbb{R})$, then x lies in the image of $H^*_{Bb}(G, A) \to H^*_B(G, A)$.*

Proof. We first consider the case of an A with underlying abelian group torsion-free. In that case, tensoring with $\mathbb{Z} \to \mathbb{R} \to S^1$ yields a short exact sequence $A \to A \otimes \mathbb{R} \to A \otimes S^1$ in $mP(G)$, which admits a Borel bounded section $A \otimes S^1 \to A \otimes \mathbb{R}$ of the underlying spaces (cf. [4]). Therefore, we obtain a commutative diagram with exact rows

and vertical maps α and δ isomorphisms, because $A \otimes S^1$ has finite diameter:

$$\begin{tikzcd}
H^{n-1}_{Bb}(G, A \otimes S^1) \arrow[r] \arrow[d, "\cong"', "\alpha"] & H^n_{Bb}(G, A) \arrow[r] \arrow[d, "\beta"] & H^n_{Bb}(G, A \otimes \mathbb{R}) \arrow[r] \arrow[d, "\gamma"] & H^n_{Bb}(G, A \otimes S^1) \arrow[d, "\cong"', "\delta"] \\
H^{n-1}_{B}(G, A \otimes S^1) \arrow[r] & H^n_{B}(G, A) \arrow[r] & H^n_{B}(G, A \otimes \mathbb{R}) \arrow[r] & H^n_{B}(G, A \otimes S^1).
\end{tikzcd}$$

The result then follows from a simple diagram chase. If A has torsion, denote by $TA < A$ the torsion subgroup (with the induced metric); it is a G-submodule. We have a $\mathbb{Z}$-split short exact sequence $TA \to A \to A/TA$ in $mP(G)$, which gives rise to a commutative diagram with exact rows and isomorphisms χ and μ, because TA is finite:

$$\begin{tikzcd}
H^n_{Bb}(G, TA) \arrow[r] \arrow[d, "\cong"', "\chi"] & H^n_{Bb}(G, A) \arrow[r] \arrow[d, "\psi"] & H^n_{Bb}(G, A/TA) \arrow[r] \arrow[d, "\phi"] & H^{n+1}_{Bb}(G, TA) \arrow[d, "\cong"', "\mu"] \\
H^n_{B}(G, TA) \arrow[r] & H^n_{B}(G, A) \arrow[r] & H^n_{B}(G, A/TA) \arrow[r] & H^{n+1}_{B}(G, TA).
\end{tikzcd}$$

Having proved the claim of the lemma for A/TA and using that $A \otimes \mathbb{R} \cong (A/TA) \otimes \mathbb{R}$, a simple diagram chase completes the proof for the general case. □

Lemma 4.2. *Let G be a virtually connected, separable, locally compact topological group, with a topology given by a complete metric and let M in $mP(G)$ be finitely generated as an abelian group and with discrete topology. Let $x \in H^*_B(G, M)$ and assume that for some $n > 0$, nx lies in the image of $H^*_{Bb}(G, M) \to H^*_B(G, M)$. Then so does x.*

Proof. The connected component G_0 acts trivially on M and has finite index in G. From our assumption we infer that the image $\overline{x}$ of x in $H^*_B(G, M \otimes \mathbb{R})$ has the property that $n\overline{x}$ lies in the image of the map $H^*_{Bb}(G, M \otimes \mathbb{R}) \to H^*_B(G, H \otimes \mathbb{R})$. Since that image is an $\mathbb{R}$-vector space, we conclude that $\overline{x}$ lies in that image too. Using Lemma 4.1, it follows that already x lies in the image of $H^*_{Bb}(G, M) \to H^*_B(G, M)$. □

The following result was proved in [1] for connected Lie groups and trivial coefficients $A = \mathbb{Z}$. However, in applications one is often confronted with Lie groups with finitely many connected components (virtually connected Lie groups), like $GL(n, \mathbb{R})$ or $O(p, q)$. The transfer map allows a natural generalization of [1, Theorem 1.2] to include such groups.

Corollary 4.3. *Let G be a virtually connected Lie group. Then the following are equivalent:*

(1) *the radical* $\sqrt{G}$ *is linear;*
(2) *for every finitely generated* G/G_0*-module* M*, the forgetful map* $H^*_{Bb}(G, M) \to H^*_B(G, M)$ *is surjective.*

Proof. That (2)⇒(1) is the content of Corollary 2.5, so we show that (1)⇒(2). For this we consider the commutative diagram

$$\begin{array}{ccccc} H^*_{Bb}(G, M) & \xrightarrow{\text{Res}} & H^*_{Bb}(G_0, M) & \xrightarrow{\text{Tran}} & H^*_{Bb}(G, M) \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \eta} \\ H^*_B(G, M)) & \xrightarrow{\text{Res}} & H^*_B(G_0, M) & \xrightarrow{\text{Tran}} & H^*_B(G, M)\,. \end{array}$$

Since G_0 is connected, the G_0-action on M is trivial. Because $\sqrt{G} = \sqrt{G_0}$ we conclude from [1, Lemma 3.5] that ζ is surjective. It follows that for $x \in H^*_B(G, M)$, the element $\mathrm{Tran} \circ \mathrm{Res(x)} = [\mathrm{G} : \mathrm{G_0}] \cdot \mathrm{x}$ lies in the image of η. By Lemma 4.2, we conclude that x lies in the image of η too, completing the proof. ☐

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Mapmo Université D'Orléans, Orléans
E-mail address: indira.chatterji@univ-orleans.fr

Mathematics Department, ETH Zürich
E-mail address: mislin@math.ethz.ch
Current address: Mathematics Department, Ohio-State University
E-mail address: mislin@math.ohio-state.edu